# BMath-II, Algebra-IV, Mid-Sem test (2019)

**Instructions :** Total time 2 hours. Maximum marks 30. You may use results (not problems) done in the class without proof. For any field $L$, we denote the set of non-zero elements in $L$ by $L^*$. All questions carry equal marks.

1. Let $L/\mathbb{Q}$ be a finite field extension. Prove that the group of elements in the multiplicative group $L^*$ having finite order, is finite. (6)

2. Let $K$ be a finite field extension of a field $k$. Assume that $gcd(6, [K:k]) = 1$. Prove that for any $u \in K$, $k(u) = k(u^2) = k(u^3)$. (6)

3. Let $k$ be a field and $f(X), g(X) \in k[X]$ be irreducible, $deg(f) = m$, $deg(g) = n$. Assume $gcd(m, n) = 1$. If $\alpha$ be a root of $f(X)$ in some extension of $k$, prove that $g(X)$ is irreducible in $k(\alpha)[X]$. (6)

4. Let $p$ and $q$ be distinct primes. Let $L$ be the splitting field of $X^p - q$ over $\mathbb{Q}$. Compute the degree $[L:\mathbb{Q}]$. (6)

5. Let $L/k$ be a finite extension of fields. Assume that $L = k(S)$, $S \subset L$, not necessarily finite, such that $s^2 \in k$ for all $s \in S$. Prove that $[L:k] = 2^r$ for some $r \geq 0$. Give an example to show that in general, in the hypothesis on $S$, we cannot replace 2 by an odd prime $p$ to conclude $[L:k] = p^r$. (6=2+4)

6. Determine all fields $K$ such that the additive group $(K, +)$ is cyclic. (6)

7. Show that the polynomials $f(X) = X^3 + X + 1$ and $g(X) = X^3 + X^2 + 1$ over $\mathbb{F}_2$ are irreducible. Let $\alpha$ be a root of $f(X)$ and $\beta$ be a root of $g(X)$ and let $K = \mathbb{F}_2(\alpha),\ L = \mathbb{F}_2(\beta)$. Give an explicit $\mathbb{F}_2$-linear field isomorphism $K \cong L$. (6=1+5)